

PROJECTIVELY GENERATED d -ABELIAN CATEGORIES ARE d -CLUSTER TILTING

SONDRE KVAMME

ABSTRACT. Building on work of Jasso, we prove that any projectively generated d -abelian category is equivalent to a d -cluster tilting subcategory of an abelian category with enough projectives. This supports the claim that d -abelian categories are good axiomatizations of d -cluster tilting subcategories.

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1. INTRODUCTION

The concept of d -cluster tilting subcategories was introduced by Iyama in [I1], and further developed in [I2], [I3]. It is the natural framework for doing higher Auslander-Reiten theory. A d -cluster tilting subcategory \mathcal{M} is a contravariantly finite, covariantly finite, and generating-cogenerating subcategory of an abelian category \mathcal{A} satisfying

$$\mathcal{M} = \{X \in \mathcal{A} \mid \forall i \in \{1, 2, \dots, d-1\} \text{ Ext}_{\mathcal{A}}^i(X, \mathcal{M}) = 0\} \quad (1.1)$$

$$\{X \in \mathcal{A} \mid \forall i \in \{1, 2, \dots, d-1\} \text{ Ext}_{\mathcal{A}}^i(\mathcal{M}, X) = 0\} \quad (1.2)$$

Examples of such categories are given in [HI1], [HI2], [IO]. A problem with this definition is that it is not clear which properties of \mathcal{M} are independent of the embedding into \mathcal{A} . To fix this, Jasso introduced in [J] the concept of a d -abelian category (see Definition 2.3), which is an axiomatization of d -cluster tilting subcategories. He shows that any d -cluster tilting subcategory is d -abelian. Furthermore, he also shows [J, Theorem 3.20] that if \mathcal{M} is a small

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projectively generated d -abelian category with category of projective objects denoted by \mathcal{P} , such that there exists an exact duality $D: \text{mod } \mathcal{P} \rightarrow \text{mod } \mathcal{P}^{\text{op}}$, then the image of the fully faithful functor

$$F: \mathcal{M} \rightarrow \text{mod } \mathcal{P} \quad F(X) = \mathcal{M}(-, X)|_{\mathcal{P}}$$

is d -cluster tilting in $\text{mod } \mathcal{P}$. Here $\text{mod } \mathcal{P}$ is the category of finitely presented contravariant functors from \mathcal{P} to $\text{Mod } \mathbb{Z}$. In this note we show that the second assumption is unnecessary.

Theorem 1.3. *Let \mathcal{M} be a small projectively generated d -abelian category, let \mathcal{P} be the set of projective objects of \mathcal{M} , and let $F: \mathcal{M} \rightarrow \text{mod } \mathcal{P}$ be the functor defined by $F(X) := \mathcal{M}(-, X)|_{\mathcal{P}}$. Then the essential image*

$$F\mathcal{M} := \{M \in \text{mod } \mathcal{P} \mid \exists X \in \mathcal{M} \text{ such that } F(X) \cong M\}$$

is d -cluster tilting in $\text{mod } \mathcal{P}$.

We emphasize that almost all of the work towards proving this theorem has been done in [J]. In fact, by Lemma 2.6 the only thing which remains is to show that $F\mathcal{M}$ is cogenerating and contravariantly finite, and the proof of these properties are straightforward.

2. PRELIMINARIES

We recall the definition of d -exact sequences and d -abelian categories.

Definition 2.1 ([J, Definition 2.2]). Let \mathcal{M} be an additive category and $f^0: X^0 \rightarrow X^1$ a morphism in \mathcal{M} . A d -cokernel of f^0 is a sequence of maps

$$(f^1, \dots, f^d): X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \dots \xrightarrow{f^{d-1}} X^d \xrightarrow{f^d} X^{d+1}$$

such that the sequence

$$\begin{aligned} 0 \rightarrow \mathcal{M}(X^{d+1}, Z) &\xrightarrow{-\circ f^d} \mathcal{M}(X^d, Z) \xrightarrow{-\circ f^{d-1}} \dots \\ &\dots \xrightarrow{-\circ f^1} \mathcal{M}(X^1, Z) \xrightarrow{-\circ f^0} \mathcal{M}(X^0, Z) \end{aligned}$$

is exact for all $Z \in \mathcal{M}$. Dually, a d -kernel of a morphism $g^d: Y^d \rightarrow Y^{d+1}$ is a sequence of maps

$$(g^0, \dots, g^{d-1}): Y^0 \xrightarrow{g^0} Y^1 \xrightarrow{g^1} \dots \xrightarrow{g^{d-2}} Y^{d-1} \xrightarrow{g^{d-1}} Y^d$$

such that the sequence

$$\begin{aligned} 0 \rightarrow \mathcal{M}(Z, Y^0) &\xrightarrow{g^0 \circ -} \mathcal{M}(Z, Y^1) \xrightarrow{g^1 \circ -} \dots \\ &\dots \xrightarrow{g^{d-1} \circ -} \mathcal{M}(Z, Y^d) \xrightarrow{g^d \circ -} \mathcal{M}(Z, Y^{d+1}) \end{aligned}$$

is exact for all $Z \in \mathcal{M}$.

Definition 2.2 ([J, Definition 2.4]). Let \mathcal{M} be an additive category. A d -exact sequence is a sequence of maps

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \dots \xrightarrow{f^d} X^{d+1}$$

such that (f^0, \dots, f^{d-1}) is a d -kernel of f^d , and (f^1, \dots, f^d) is a d -cokernel of f^0 .

Recall that \mathcal{M} is *idempotent complete* if for any idempotent $e: X \rightarrow X$ in \mathcal{M} there exists morphisms $\pi: X \rightarrow Y$ and $i: Y \rightarrow X$ such that $i \circ \pi = e$ and $\pi \circ i = 1_Y$.

Definition 2.3 ([J, Definition 3.1]). A d -abelian category is an additive category \mathcal{M} satisfying the following axioms:

- (A0) \mathcal{M} is idempotent complete.
- (A1) Every morphism in \mathcal{M} has a d -kernel and a d -cokernel
- (A2) Let $f^0: X^0 \rightarrow X^1$ be a monomorphism and (f^1, \dots, f^d) a d -cokernel of f^0 . Then the sequence

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \dots \xrightarrow{f^{d-1}} X^d \xrightarrow{f^d} X^{d+1}$$

is d -exact.

- (A2^{op}) Let $g^d: Y^d \rightarrow Y^{d+1}$ be an epimorphism and (g^0, \dots, g^{d-1}) a d -kernel of g^d . Then the sequence

$$Y^0 \xrightarrow{g^0} Y^1 \xrightarrow{g^1} \dots \xrightarrow{g^{d-2}} Y^{d-1} \xrightarrow{g^{d-1}} Y^d \xrightarrow{g^d} Y^{d+1}$$

is d -exact.

Recall that $P \in \mathcal{M}$ is projective if for every epimorphism $f: X \rightarrow Y$ in \mathcal{M} the sequence $\mathcal{M}(P, X) \xrightarrow{f \circ -} \mathcal{M}(P, Y) \rightarrow 0$ is exact. The following results holds for projective objects in d -abelian categories.

Theorem 2.4 ([J, Theorem 3.12]). Let \mathcal{M} be a d -abelian category, let P be a projective object in \mathcal{M} , let $f^0: X^0 \rightarrow X^1$ be a morphism in \mathcal{M} , and let (f^1, \dots, f^d) be a d -cokernel of f^0 . Then the sequence

$$\begin{aligned} \mathcal{M}(P, X^0) \xrightarrow{f^0 \circ -} \mathcal{M}(P, X^1) \xrightarrow{f^1 \circ -} \mathcal{M}(P, X^2) \xrightarrow{f^2 \circ -} \dots \\ \dots \xrightarrow{f^d \circ -} \mathcal{M}(P, X^{d+1}) \rightarrow 0 \end{aligned}$$

is exact.

Definition 2.5 ([J, Definition 3.19]). Let \mathcal{M} be a d -abelian category. We say that \mathcal{M} is *projectively generated* if for every objects $X \in \mathcal{M}$ there exists a projective object $P \in \mathcal{M}$ and an epimorphism $f: P \rightarrow X$.

Let \mathcal{M} be a projectively generated d -abelian category, let \mathcal{P} be the category of projective objects of \mathcal{M} , and let $F: \mathcal{M} \rightarrow \text{mod } \mathcal{P}$ be the functor

$F(X) = \mathcal{M}(-, X)|_{\mathcal{P}}$. Theorem 2.4 tells us that if (f^1, \dots, f^d) is a d -cokernel of f^0 , then the sequence

$$F(X^0) \xrightarrow{F(f^0)} F(X^1) \xrightarrow{F(f^1)} F(X^2) \xrightarrow{F(f^2)} \dots \xrightarrow{F(f^d)} F(X^{d+1}) \rightarrow 0$$

is exact in $\text{mod } \mathcal{P}$.

Parts of the proof that a projectively generated d -abelian category is d -cluster tilting in $\text{mod } \mathcal{P}$ follows from the following lemma. Note that there is a typo in [J]; in the lemma they write that $F\mathcal{M}$ is contravariantly finite, but in the proof they show that it is covariantly finite.

Lemma 2.6 ([J, Lemma 3.22]). *Let \mathcal{M} be a small projectively generated d -abelian category, let \mathcal{P} the category of projective objects of \mathcal{M} , and let $F: \mathcal{M} \rightarrow \text{mod } \mathcal{P}$ be the functor defined by $F(X) = \mathcal{M}(-, X)|_{\mathcal{P}}$. Also, let*

$$F\mathcal{M} := \{M \in \text{mod } \mathcal{P} \mid \exists X \in \mathcal{M} \text{ such that } F(X) \cong M\}$$

be the essential image of F . Then the following holds:

- (i) $\text{mod } \mathcal{P}$ is abelian;
- (ii) F is fully faithful;
- (iii) For all $k \in \{1, \dots, d-1\}$ we have

$$\text{Ext}_{\text{mod } \mathcal{P}}^k(F\mathcal{M}, F\mathcal{M}) = 0;$$

- (iv) We have

$$F\mathcal{M} = \{M \in \text{mod } \mathcal{P} \mid \forall i \in \{1, 2, \dots, d-1\} \text{ Ext}_{\text{mod } \mathcal{P}}^i(M, F\mathcal{M}) = 0\};$$

- (v) We have

$$F\mathcal{M} = \{M \in \text{mod } \mathcal{P} \mid \forall i \in \{1, 2, \dots, d-1\} \text{ Ext}_{\text{mod } \mathcal{P}}^i(F\mathcal{M}, M) = 0\};$$

- (vi) $F\mathcal{M}$ is covariantly finite in $\text{mod } \mathcal{P}$.

Since $F\mathcal{M}$ is obviously generating, it only remains to show that $F\mathcal{M}$ is cogenerating and contravariantly finite.

3. PROOF OF THEOREM 1.3

Throughout this section we fix an integer $d \geq 2$, a projectively generated d -abelian category \mathcal{M} , and we let \mathcal{P} denote the category of projective objects in \mathcal{M} .

Lemma 3.1. *$F\mathcal{M}$ is cogenerating in $\text{mod } \mathcal{P}$.*

Proof. Let $G \in \text{mod } \mathcal{P}$ be arbitrary. Since G is finitely presented, we can find projective objects $P^0, P^1 \in \mathcal{M}$ and a morphism $\phi: F(P^0) \rightarrow F(P^1)$ such that $\text{Cok } \phi \cong G$. Since F is full, there exists a morphism $f^0: P^0 \rightarrow P^1$ in \mathcal{M} such that $F(f^0) = \phi$. Let

$$(f^1, \dots, f^d): P^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \dots \xrightarrow{f^{d-1}} X^d \xrightarrow{f^d} X^{d+1}$$

be a d -cokernel of f^0 . By Theorem 2.4 we know that the sequence

$$F(P^0) \xrightarrow{F(f^0)} F(P^1) \xrightarrow{F(f^1)} F(X^2) \xrightarrow{F(f^2)} \dots \xrightarrow{F(f^d)} F(X^{d+1}) \rightarrow 0$$

is exact. In particular, we have a monomorphism

$$G \cong \text{Cok}(F(f^0)) \rightarrow F(X^2)$$

This shows that $F\mathcal{M}$ is cogenerating. \square

Lemma 3.2. *$F\mathcal{M}$ is contravariantly finite in $\text{mod } \mathcal{P}$.*

Proof. Let $G \in \text{mod } \mathcal{P}$ be arbitrary. By Lemma 3.1 there exist objects $X^d, X^{d+1} \in \mathcal{M}$ and an exact sequence

$$0 \rightarrow G \xrightarrow{i} F(X^d) \xrightarrow{\phi} F(X^{d+1})$$

where $\phi = F(f^d)$ since F is full. Let

$$(f^0, \dots, f^{d-1}) : X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \dots \xrightarrow{f^{d-2}} X^{d-1} \xrightarrow{f^{d-1}} X^d$$

be a d -kernel of f^d . Since $F(f^d) \circ F(f^{d-1}) = 0$, we get an induced morphism $F(X^{d-1}) \xrightarrow{p} G$. We claim that p is a right $F\mathcal{M}$ -approximation of G . Let $X \in \mathcal{M}$ and let $F(X) \xrightarrow{\psi} G$ be an arbitrary morphism in $\text{mod } \mathcal{P}$. Since F is full, the composition $F(X) \xrightarrow{\psi} G \xrightarrow{i} F(X^d)$ is of the form $F(f)$ for some morphism $f : X \rightarrow X^d$. Since $f^d \circ f = 0$ and

$$\mathcal{M}(X, X^{d-1}) \xrightarrow{f^{d-1} \circ -} \mathcal{M}(X, X^d) \xrightarrow{f^d \circ -} \mathcal{M}(X, X^{d+1})$$

is exact, it follows that $f = f^{d-1} \circ g$ for some morphism $g : X \rightarrow X^{d-1}$. Applying F gives

$$i \circ p \circ F(g) = F(f^{d-1}) \circ F(g) = F(f) = i \circ \psi$$

and since i is a monomorphism, we get that $p \circ F(g) = \psi$. This shows that p is a right $F\mathcal{M}$ -approximation, and since G was arbitrary it follows that $F\mathcal{M}$ is contravariantly finite. \square

Remark 3.3. Let \mathcal{M} be an injectively cogenerated d -abelian category, and let \mathcal{I} be the category of injective objects in \mathcal{M} . Furthermore, let $G : \mathcal{M} \rightarrow (\mathcal{I} \text{ mod})^{\text{op}}$ be the functor given by $G(X) := \mathcal{M}(X, -)|_{\mathcal{I}}$. Here $\mathcal{I} \text{ mod}$ denotes the category of finitely presented covariant functors from \mathcal{I} to $\text{mod } \mathbb{Z}$. The dual of Theorem 1.3 tells us that G is a fully faithful functor, $(\mathcal{I} \text{ mod})^{\text{op}}$ is an abelian category, and the essential image

$$G\mathcal{M} := \{M \in (\mathcal{I} \text{ mod})^{\text{op}} \mid \exists X \in \mathcal{M} \text{ such that } G(X) \cong M\}$$

is d -cluster tilting in $(\mathcal{I} \text{ mod})^{\text{op}}$.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, GERMANY
E-mail address: `sondre@math.uni-bonn.de`